Sea $L: \mathbb{R}^4 \to \mathbb{R}^4$ definida como:

$$L\left(\begin{bmatrix} x \\ y \\ z \\ w \end{bmatrix}\right) = \begin{bmatrix} 1 & 2 & 1 & 3 \\ 2 & 1 & -1 & 2 \\ 1 & 0 & 0 & -1 \\ 4 & 1 & -1 & 0 \end{bmatrix} \begin{bmatrix} x \\ y \\ z \\ w \end{bmatrix}$$

a) Determine una base para núcleo (L).

Núcleo:

$$\left[\begin{array}{cccc|c} 1 & 2 & 1 & 3 & 0 \\ 2 & 1 & -1 & 2 & 0 \\ 1 & 0 & 0 & -1 & 0 \\ 4 & 1 & -1 & 0 & 0 \end{array}\right] \begin{array}{l} R_2 - 2R_1 \to R_2 \\ R_3 - R_1 \to R_3 \\ R_4 - 4R_1 \to R_4 \\ \sim \end{array} \left[\begin{array}{cccc|c} 1 & 2 & 1 & 3 & 0 \\ 0 & -3 & -3 & -4 & 0 \\ 0 & -2 & -1 & -4 & 0 \\ 0 & -7 & -5 & -12 & 0 \end{array}\right]$$

$$\begin{array}{l} R_2 \cdot 2 \to R_2 \\ \sim \\ R_3 \cdot 3 \to R_3 \end{array} \left[\begin{array}{cccc|c} 1 & 2 & 1 & 3 & 0 \\ 0 & -6 & -6 & -8 & 0 \\ 0 & -6 & -3 & -12 & 0 \\ 0 & -7 & -5 & -12 & 0 \end{array}\right] \begin{array}{l} R_3 + R_2 \to R_3 \\ \sim \end{array}$$

$$\left[\begin{array}{cccc|c} 1 & 2 & 1 & 3 & 0 \\ 0 & -6 & -6 & -8 & 0 \\ 0 & 0 & 3 & -4 & 0 \\ 0 & -7 & -5 & -12 & 0 \end{array}\right] \begin{array}{l} R_2 \cdot \left(-\frac{1}{6}\right) \to R_2 \\ \sim \end{array} \left[\begin{array}{cccc|c} 1 & 2 & 1 & 3 & 0 \\ 0 & 1 & 1 & \frac{4}{3} & 0 \\ 0 & 0 & 3 & -4 & 0 \\ 0 & -7 & -5 & -12 & 0 \end{array}\right]$$

$$\begin{array}{l} R_4 + 7R_2 \to R_4 \\ \sim \end{array} \left[\begin{array}{cccc|c} 1 & 2 & 1 & 3 & 0 \\ 0 & 1 & 1 & \frac{4}{3} & 0 \\ 0 & 0 & 3 & -4 & 0 \\ 0 & 0 & 2 & -\frac{8}{3} & 0 \end{array}\right] \begin{array}{l} R_4 \cdot 3 \to R_4 \\ \sim \end{array} \left[\begin{array}{cccc|c} 1 & 2 & 1 & 3 & 0 \\ 0 & 1 & 1 & \frac{4}{3} & 0 \\ 0 & 0 & 3 & -4 & 0 \\ 0 & 0 & 6 & -8 & 0 \end{array}\right]$$

$$\begin{array}{l} R_4 - 2R_3 \to R_4 \\ \sim \end{array} \left[\begin{array}{cccc|c} 1 & 2 & 1 & 3 & 0 \\ 0 & 1 & 1 & \frac{4}{3} & 0 \\ 0 & 0 & 3 & -4 & 0 \\ 0 & 0 & 0 & 0 & 0 \end{array}\right] \begin{array}{l} R_2 \cdot 3 \to R_2 \\ \sim \end{array} \left[\begin{array}{cccc|c} 1 & 2 & 1 & 3 & 0 \\ 0 & 3 & 3 & 4 & 0 \\ 0 & 0 & 3 & -4 & 0 \\ 0 & 0 & 0 & 0 & 0 \end{array}\right]$$

entonces tenemos que:

(1) $x + 2y + z + 3w = 0$
(2) $3y + 3z + 4w = 0$
(3) $3z - 4w = 0 \Rightarrow$

Usamos el parámetro t con $w = t$.

(1)
$$3z - 4t = 0$$
$$3z = 4t$$
$$z = \frac{4}{3}t$$

Sustituimos en (2)

$$3y + 3\left(\frac{4}{3}t\right) + 4t = 0$$
$$3y + 4t + 4t = 0$$
$$3y + 8t = 0$$
$$3y = -8t$$
$$y = -\frac{8}{3}t$$

Sustituimos en (1)

$$x + 2\left(\frac{-8}{3}t\right) + \frac{4}{3}t + 3t = 0$$
$$x - \frac{16}{3}t + \frac{4}{3}t + 3t = 0$$
$$x - \frac{12}{3}t + 3t = 0$$
$$x - 4t + 3t = 0$$
$$x - t = 0$$
$$\underline{\underline{x = t}}\,//$$

$\therefore$
$$x = t$$
$$y = -\frac{8}{3}t$$
$$z = \frac{4}{3}t$$
$$w = t$$

Si $(x, y, z, w) \in \text{Ker } L$, (x, y, z, w)

$$= \left(t, -\frac{8}{3}t, \frac{4}{3}t, t\right) = t\left(1, -\frac{8}{3}, \frac{4}{3}, 1\right)$$

$$\Rightarrow \text{Ker } L = \left\{\left(1, -\frac{8}{3}, \frac{4}{3}, 1\right)\right\} = \{(3, -8, 4, 3)\}$$

$\therefore \{(3, -8, 4, 3)\}$ base de Ker L dim (1)

(b) Determine una base para Im(L).

$$\left[\begin{array}{cccc|c} 1 & 2 & 1 & 3 & a \\ 2 & 1 & -1 & 2 & b \\ 1 & 0 & 0 & -1 & c \\ 4 & 1 & -1 & 0 & d \end{array}\right] \begin{array}{l} R_2 - 2R_1 \to R_2 \\ R_3 - R_1 \to R_3 \\ R_4 - 4R_1 \to R_4 \\ \sim \end{array} \left[\begin{array}{cccc|c} 1 & 2 & 1 & 3 & a \\ 0 & -3 & -3 & -4 & b-2a \\ 0 & -2 & -1 & -4 & c-a \\ 0 & -7 & -5 & -12 & d-4a \end{array}\right]$$

$$\begin{array}{l} R_2 \cdot 2 \to R_2 \\ R_3 \cdot 3 \to R_3 \\ \sim \end{array} \left[\begin{array}{cccc|c} 1 & 2 & 1 & 3 & a \\ 0 & -6 & -6 & -8 & 2b-4a \\ 0 & -6 & -3 & -12 & 3c-3a \\ 0 & -7 & -5 & -12 & d-4a \end{array}\right] \begin{array}{l} R_3 - R_2 \to R_3 \\ \sim \end{array}$$

$$\left[\begin{array}{cccc|c} 1 & 2 & 1 & 3 & a \\ 0 & -6 & -6 & -8 & 2b-4a \\ 0 & 0 & 3 & -4 & 3c-2b+a \\ 0 & -7 & -5 & -12 & d-4a \end{array}\right] \begin{array}{l} R_2\left(-\frac{1}{6}\right) \to R_2 \\ \sim \end{array}$$

$$\left[\begin{array}{cccc|c} 1 & 2 & 1 & 3 & a \\ 0 & 1 & 1 & \frac{4}{3} & -\frac{b-2a}{3} \\ 0 & 0 & 3 & -4 & 3c-2b+a \\ 0 & -7 & -5 & -12 & d-4a \end{array}\right] \begin{array}{l} R_4 + 7R_2 \to R_4 \\ \sim \end{array}$$

$$\left[\begin{array}{cccc:c} 1 & 2 & 1 & 3 & a \\ 0 & 1 & 1 & \frac{4}{3} & -\frac{b-2a}{3} \\ 0 & 0 & 3 & -4 & 3c-2b+a \\ 0 & 0 & 2 & -\frac{8}{3} & \frac{2a+3d-7b}{3} \end{array}\right] \quad R_4 \cdot 3 \to R_4 \quad \sim$$

$$\left[\begin{array}{cccc:c} 1 & 2 & 1 & 3 & a \\ 0 & 1 & 1 & \frac{4}{3} & -\frac{b-2a}{3} \\ 0 & 0 & 3 & -4 & 3c-2b+a \\ 0 & 0 & 6 & -8 & 2a+3d-7b \end{array}\right] \quad R_4 - 2R_3 \to R_4 \quad \sim$$

$$\left[\begin{array}{cccc:c} 1 & 2 & 1 & 3 & a \\ 0 & 1 & 1 & \frac{4}{3} & -\frac{b-2a}{3} \\ 0 & 0 & 3 & -4 & 3c-2b+a \\ 0 & 0 & 0 & 0 & 3d-3b-6c \end{array}\right]$$

$\Rightarrow 3d-3b-6c=0$ si $a=x$, $b=y$, $c=z$, $d=w$

$\Rightarrow 3w-3y-6z=0 \Rightarrow w-y-2z=0$

$(-y-2z+w=0)(-1)$

$= y+2z-w=0$

$\therefore \operatorname{Im}(T) = \{(x,y,z,w) \in \mathbb{R}^4 \mid y+2z-w=0\}$

$\dim = 3$

(c) Verifique el teorema 10.7

TEOREMA (10.7)

Si $L: V \to W$ es una transformación lineal de un espacio vectorial V, de dimensión n, en un espacio vectorial W, entonces

$$\text{nulidad}(L) + \text{rango}(L) = \dim V.$$

Esto es:

$$\dim(\text{Ker}(L)) + \dim(\text{Im}(L)) = \dim \mathbb{R}^4$$
$$1 + 3 = 4$$
$$4 = 4$$

$\therefore$ Se cumple