Ejercicio 4.3

Muestra si el siguiente conjunto es base para $\mathbb{R}^4$

4.3

$$\left\{ \begin{bmatrix} 0 \\ 0 \\ 1 \\ 1 \end{bmatrix}, \begin{bmatrix} -1 \\ 1 \\ 1 \\ 2 \end{bmatrix}, \begin{bmatrix} 1 \\ 1 \\ 0 \\ 0 \end{bmatrix}, \begin{bmatrix} 2 \\ 1 \\ 2 \\ 1 \end{bmatrix} \right\} \subset \mathbb{R}^4$$

Sean $\alpha_1, \alpha_2, \alpha_3, \alpha_4 \in \mathbb{R}$, Primero comprobamos la independencia lineal, Esto es:

$$\alpha_1 \begin{bmatrix} 0 \\ 0 \\ 1 \\ 1 \end{bmatrix} + \alpha_2 \begin{bmatrix} -1 \\ 1 \\ 1 \\ 2 \end{bmatrix} + \alpha_3 \begin{bmatrix} 1 \\ 1 \\ 0 \\ 0 \end{bmatrix} + \alpha_4 \begin{bmatrix} 2 \\ 1 \\ 2 \\ 1 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \\ 0 \end{bmatrix}$$

Tomamos la matriz de coeficientes

$$\left[\begin{array}{cccc:c} 0 & -1 & 1 & 2 & 0 \\ 0 & 1 & 1 & 1 & 0 \\ 1 & 1 & 0 & 2 & 0 \\ 1 & 2 & 0 & 1 & 0 \end{array}\right]$$

$(R_4)(-2)+R_3 \longrightarrow R_3$,

$$\left[\begin{array}{cccc|c} 0 & -1 & 1 & 2 & 0 \\ 0 & 1 & 1 & 1 & 0 \\ -1 & -3 & 0 & 0 & 0 \\ 1 & 2 & 0 & 1 & 0 \end{array}\right]$$

$R_3 + R_4 \longrightarrow R_4$, $R_2 + R_1 \longrightarrow R_1$

$$\left[\begin{array}{cccc|c} 0 & 0 & 2 & 3 & 0 \\ 0 & 1 & 1 & 1 & 0 \\ -1 & -3 & 0 & 0 & 0 \\ 0 & -1 & 0 & 1 & 0 \end{array}\right]$$

$R_4 + R_2 \longrightarrow R_2$

$$\left[\begin{array}{cccc|c} 0 & 0 & 2 & 3 & 0 \\ 0 & 0 & 1 & 2 & 0 \\ -1 & -3 & 0 & 0 & 0 \\ 0 & -1 & 0 & 1 & 0 \end{array}\right]$$

$(R_2)(-2)+R_1 \longrightarrow R_1$

$$\left[\begin{array}{cccc|c} 0 & 0 & 0 & -1 & 0 \\ 0 & 0 & 1 & 2 & 0 \\ -1 & -3 & 0 & 0 & 0 \\ 0 & -1 & 0 & 1 & 0 \end{array}\right]$$

Determinante:

$$-1\left[-1\begin{bmatrix} 0 & -1 \\ 1 & 2 \end{bmatrix}\right]$$

$$= -1\left[-1((0)(2)-(1)(-1))\right]$$

$$= -1\left[-1(1)\right] = \underline{1}$$

$$\alpha_4 = 0 \quad (1)$$

$$\alpha_3 + 2\alpha_4 = 0 \quad (2)$$

$$-\alpha_1 - 3\alpha_2 + 0 + 0 = 0 \quad (3)$$

$$0 - \alpha_2 + 0 + \alpha_4 = 0 \quad (4)$$

$\Rightarrow \alpha_4 = 0$ Sustituyendo en (2) y (4)

$\alpha_4 = 0,\ \alpha_2 = 0$, Sustituyendo en (3) $\alpha_1 = 0$

La solución es trivial: $\alpha_1 = \alpha_2 = \alpha_3 = \alpha_4 = 0$

$\therefore$ El conjunto es linearmente independiente

Ahora veamos si genera a $\mathbb{R}^4$

$$\left[\begin{array}{cccc|c} 0 & -1 & 1 & 2 & w \\ 0 & 1 & 1 & 1 & x \\ 1 & 1 & 0 & 2 & y \\ 1 & 2 & 0 & 1 & z \end{array}\right]$$

$R_2 + R_1 \longrightarrow R_1$, $(R_4)(-1) + R_3 \longrightarrow R_3$

$$\left[\begin{array}{cccc|c} 0 & 0 & 2 & 3 & w+x \\ 0 & 1 & 1 & 1 & x \\ 0 & -1 & 0 & 1 & y-z \\ 1 & 2 & 0 & 1 & z \end{array}\right]$$

$R_2 + R_3 \longrightarrow R_3$

$$\left[\begin{array}{cccc|c} 0 & 0 & 2 & 3 & w+x \\ 0 & 1 & 1 & 1 & x \\ 0 & 0 & 1 & 2 & x+y-z \\ 1 & 2 & 0 & 1 & z \end{array}\right]$$

$(R_3)(-2)+R_1 \longrightarrow R_1$

$$\left[\begin{array}{cccc|c} 0 & 0 & 0 & -1 & -x-2y+w+2z \\ 0 & 1 & 1 & 1 & x \\ 0 & 0 & 1 & 2 & x+y-z \\ 1 & 2 & 0 & 1 & z \end{array}\right]$$

$R_1+R_4 \longrightarrow R_4$, $(R_3)(-1)+R_2 \longrightarrow R_2$

$$\left[\begin{array}{cccc|c} 0 & 0 & 0 & -1 & w-x-2y+2z \\ 0 & 1 & 0 & -1 & -y+z \\ 0 & 0 & 1 & 2 & x+y-z \\ 1 & 2 & 0 & 0 & -x-2y+w+3z \end{array}\right]$$

$(R_1)(-1)+R_2 \longrightarrow R_2$, $(R_1)(2)+R_3 \longrightarrow R_3$

$$\left[\begin{array}{cccc|c} 0 & 0 & 0 & -1 & w-x-2y+2z \\ 0 & 1 & 0 & 0 & -w+x+y-z \\ 0 & 0 & 1 & 0 & 2w-x-3y+3z \\ 1 & 2 & 0 & 0 & -x-2y+w+3z \end{array}\right]$$

$(R_2)(-2)$

$$\left[\begin{array}{cc} 0 & 0 \\ 0 & 1 \\ 0 & 0 \\ 1 & 0 \end{array}\right.$$

$(R_1)(-1)$

$$\left[\begin{array}{cc} 1 & 0 \\ 0 & 1 \\ 0 & 0 \\ 0 & 0 \end{array}\right.$$

∴ El co

$(R_2)(-2)+R_4 \longrightarrow R_4$

$$\left[\begin{array}{cccc|c} 0 & 0 & 0 & -1 & w-x-2y+2z \\ 0 & 1 & 0 & 0 & -w+x+y-z \\ 0 & 0 & 1 & 0 & 2w-x-3y+3z \\ 1 & 0 & 0 & 0 & 3w-3x-4y+5z \end{array}\right]$$

$(R_1)(-1) \longrightarrow R_4,\ R_4 \longrightarrow R_1$

$$\left[\begin{array}{cccc|c} 1 & 0 & 0 & 0 & 3w-3x-4y+5z \\ 0 & 1 & 0 & 0 & -w+x+y-z \\ 0 & 0 & 1 & 0 & 2w-x-3y+3z \\ 0 & 0 & 0 & 1 & -w+x+2y-2z \end{array}\right]$$

$\therefore$ El conjunto es una base para $\mathbb{R}^4$